Math 143 (Calculus III) Theory Review Notes

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Definition [**Parametric curves and equations**] A **parametric curve** C is a two-dimensional curve whose x and y points are defined by a **pair** of functions

$$x = f(t), \quad y = g(t), \quad t \in I$$

called **parametric equations** with t usually standing for time. If $t \in [a, b]$, then the points $(f(a), g(a))$ and $(f(b), g(b))$ correspond to the **initial** and **terminal points** of the curve, respectively.

Definition [**Tangents, Areas, and Lengths**] Let $x = f(t)$ and $y = g(t)$ be a parametrization of a curve C described by $y = F(x)$. Then:

- The **slope of the tangent line** to the curve is $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$.
- Similarly, $\dfrac{d^2y}{dx^2} = \dfrac{d}{dt}\left(\dfrac{dy/dt}{dx/dt}\right)$.
- The area under $y = F(x)$ is given by
$$A = \int_a^b g(t) f'(t)\, dt, \quad x(a) = \alpha, \quad x(b) = \beta.$$
- The **length** (or **arclength**) of the curve C is given by
$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

Definition [**Polar coordinates, slope, polar area and length formulas**]

- Given $P(r, \theta)$, we can find $P(x, y)$ via $x = r\cos\theta$ and $y = r\sin\theta$.
- Given $P(x, y)$, we can find $P(r, \theta)$ via $r = \sqrt{x^2 + y^2}$ and $\theta = \arctan(y/x)$.
- Let $r = f(\theta)$ be a polar equation with parametric equations:
$$x = r\cos\theta = f(\theta)\cos\theta, \quad y = r\sin\theta = f(\theta)\sin\theta.$$
Then the **slope** at (r, θ) is given by
$$\left.\frac{dy}{dx}\right|_{(r,\theta)} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}.$$
- The **area** of $r = f(\theta)$ between $\theta = a$ and $\theta = b$ is given by
$$A = \frac{1}{2}\int_a^b [f(\theta)]^2\, d\theta.$$
- The **length** of a polar curve is given by
$$L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$
- **Remarks for graphing polar curves:**
  1. The graph with $r = a$ is a circle with radius $|a|$ centered at $(0, 0)$.
  2. The graph with polar equation $\theta = \theta_0$ is a line through $(0, 0)$ making an angle θ_0 with the x-axis.
  3. The equation for a **circle**: $r = a\sin\theta$ or $r = a\cos\theta$.
  4. The equations for **flowers**: $r = a\cos(n\theta)$ or $r = a\sin(n\theta)$. Note that if n is odd, we have n petals; if n is even we have $2n$ petals.
  5. The equations for the **cardioid**: $r = a + b\cos\theta$ or $r = a + b\sin\theta$.

Definition [**Sequence**] A **sequence** is a list of ordered numbers:

$$a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$$

Definition [**Limit Test for convergence/divergence of sequences**] A sequence $\{a_n\}$ has the **limit** L:

$$\lim_{n\to\infty} a_n = L \Rightarrow a_n \to L, \quad \text{as} \quad n \to \infty,$$

if we can make the terms a_n as close to L as $n \gg 1$. If the limit exists, we say that the sequence **converges** (or is **convergent**). Otherwise the sequence **diverges**.

Theorem [**Limit Rules**] If $\{a_n\}$ and $\{b_n\}$ are **convergent** sequences, and $c =$ const., then

- $\lim\limits_{n\to\infty}(a_n \pm b_n) = \lim\limits_{n\to\infty} a_n \pm \lim\limits_{n\to\infty} b_n$
- $\lim\limits_{n\to\infty} c\, a_n = c \lim\limits_{n\to\infty} a_n \quad \left(\lim\limits_{n\to\infty} c = c\right)$
- $\lim\limits_{n\to\infty} a_n b_n = \lim\limits_{n\to\infty} a_n \cdot \lim\limits_{n\to\infty} b_n$
- $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n}$ if $\lim\limits_{n\to\infty} b_n \neq 0$
- $\lim\limits_{n\to\infty} a_n^p = \left[\lim\limits_{n\to\infty} a_n\right]^p$ if $p > 0$ and $a_n > 0$.

Theorem [**Sandwich Theorem**] Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$. If

$$a_n \leq b_n \leq c_n \text{ for } \forall n \geq n_0 \text{ and } \lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L, \text{ then } \lim_{n\to\infty} b_n = L.$$

Theorem If $\lim\limits_{n\to\infty} a_n = L$ and the function f is continuous at L, then

$$\lim_{n\to\infty} f(a_n) = f(L).$$

Theorem Let $f(x)$ be defined $\forall x \geq n_0$ and $\{a_n\}$ such that $a_n = f(n)$ for $n \geq n_0$. Then

$$\lim_{x\to\infty} f(x) = L \Rightarrow \lim_{n\to\infty} a_n = L.$$

Theorem The above sequences converge to the limits indicated below:

- $\lim\limits_{n\to\infty} \dfrac{\ln n}{n} = 0.$
- $\lim\limits_{n\to\infty} x^{1/n} = 1, \forall x > 0.$
- $\lim\limits_{n\to\infty} \left(1 + \dfrac{x}{n}\right)^n = e^x, \forall x.$
- $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1.$
- $\lim\limits_{n\to\infty} x^n = 0$ if $|x| \leq 1.$
- $\lim\limits_{n\to\infty} \dfrac{x^n}{n!} = 0, \forall x.$

Theorem [**Monotonicity of sequences**]

- A sequence $\{a_n\}$ is called **increasing** (non-decreasing) if
$$a_n < a_{n+1} \text{ holds } \forall n \geq 1.$$
- A sequence $\{a_n\}$ is called **decreasing** (non-increasing) if
$$a_n > a_{n+1} \text{ holds } \forall n \geq 1.$$
- A sequence $\{a_n\}$ is called **monotonic** if it is either increasing or decreasing.

Theorem [**Monotonic Sequence Theorem**] Every bounded **and** monotonic sequence **is** convergent.

Definition [**Infinite series, partial sums and convergence**]

- Given $\{a_n\}$, an expression of the form
$$a_1 + a_2 + a_3 + \cdots + a_n + \ldots$$
is an **infinite series**.
- The nth **partial sum** is the sum of the first n terms:
$$s_n \doteq a_1 + a_2 + a_3 + \cdots + a_n$$
In addition, $\{s_n\}$ is the **sequence of partial sums**.
- If $s_n \to s$, the series **converges** and that its sum is s:
$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots = \sum_{n=1}^{\infty} a_n = s.$$
- On the contrary, if $\{s_n\}$ does not converge, the series **diverges**.

Definition [**Geometric series**]

- Geometric series is a series of the form of $a+ar+ar^2+ar^3+\cdots+ar^{n-1}+\cdots \doteq \sum_{n=1}^{\infty} ar^{n-1} \left(\doteq \sum_{n=0}^{\infty} ar^n\right)$, where $a \in \mathbb{R}$ is called the **leading term** and $r \in \mathbb{R}$ is the **ratio**.
- A geometric series **converges** if $|r|<1$. Its sum is given by

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \qquad (|r|<1).$$

- If $|r| \geq 1$ the geometric series **diverges**.
- Its nth partial sum, s_n, is given by

$$s_n = \frac{a\left(1-r^n\right)}{1-r}, \quad r \neq 1.$$

Definition [**Telescoping series**] A **telescoping series** is any series where nearly every term cancels with a preceding or following term.

Theorem [**The nth-term test**] If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$ (or $a_n \to 0$). On the contrary, $\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{n\to\infty} a_n$ fails to exist or if $\lim_{n\to\infty} a_n \neq 0$.

Theorem [**Combining series**] If $\sum a_n$ and $\sum b_n$ are convergent series, then so are the series $\sum c\, a_n$ (where $c=$ const.), and $\sum(a_n \pm b_n)$, and

- $\sum_{n=1}^{\infty} c\, a_n = c \sum_{n=1}^{\infty} a_n$.
- $\sum_{n=1}^{\infty} (a_n \pm b_n) = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n$.

Theorem [**The integral test**]
Suppose $f(x)$ is a continuous, positive function where $f'(x)<0$ (i.e., decreasing) on $[N,\infty)$ with $N>0$, and let $a_n = f(n)$. Then the series $\sum_{n=N}^{\infty} a_n$ and the integral $\int_N^{\infty} f(x)\,dx$ both converge **or** both diverge.

Theorem [**The p-test for series**] The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$:

- **Converges** if $p>1$.
- **Diverges** if $p \leq 1$.

Theorem [**Remainder Estimate for the Integral Test**] Suppose $f(k)=a_k$ where f is a continuous, positive, and decreasing function for $x \geq n$, and $\sum a_n$ is convergent to s. If $R_n = s - s_n$, then

$$\int_{n+1}^{\infty} f(x)dx \leq R_n \leq \int_n^{\infty} f(x)dx.$$

The value of the sum s satisfies:

$$s_n + \int_{n+1}^{\infty} f(x)dx \leq s \leq s_n + \int_n^{\infty} f(x)dx.$$

Theorem [**The Direct Comparison Test (DCT)**] Suppose that $\sum a_n$ and $\sum b_n$ are series with $a_n, b_n > 0$ (i.e., non-negative terms) and let N being some fixed integer. Then, the following two cases are discerned:

- If $\sum b_n$ is convergent and $a_n \leq b_n$, $\forall n \geq N$, then $\sum a_n$ is also convergent.
- If $\sum b_n$ is divergent and $a_n \geq b_n$, $\forall n \geq N$, then $\sum a_n$ is also divergent.

Theorem [**The Limit Comparison Test (LCT)**] Suppose $a_n, b_n > 0$, $\forall n \geq N$ for some fixed integer N.

- If $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$, then $\sum a_n$ and $\sum b_n$ both converge or both diverge.
- If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ **converges**, then $\sum a_n$ **converges**.
- If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ **diverges**, then $\sum a_n$ **diverges**.

Theorem [**Alternating series test (AST)**] If the alternating series:

$$\sum_{n=1}^{\infty} (-1)^{n+1} b_n = b_1 - b_2 + b_3 - b_4 + \ldots, \quad b_n > 0 \quad \forall n,$$

satisfies

1. $b_{n+1} \leq b_n$, $\forall n \geq N$ (for some N)
2. $b_n \to 0$ as $n \to \infty$

then the series is convergent.

Theorem [**Alternating series estimation theorem**] If the alternating series converges, then for $n \geq N$

$$s_n = b_1 - b_2 + \cdots + (-1)^{n+1} b_n,$$

approximates the sum s of the series with an error whose absolute value is less than b_{n+1}, the absolute value of the first unused term (i.e., $|a_{n+1}| = b_n$). Furthermore, the sum s lies between any two successive partial sums s_n and s_{n+1} and the remainder, $s - s_n$ has the same sign as the first unused term. In other words, we write

$$|R_n| = |s - s_n| \leq b_{n+1}.$$

Definition [**Absolute and Conditional convergence**]

- $\sum a_n$ is **absolutely convergent** if $\sum |a_n|$ is convergent.
- $\sum a_n$ is **conditionally convergent** if it is convergent but **not absolutely convergent**.
- If $\sum a_n$ **is absolutely convergent**, then $\sum a_n$ **is convergent**.

Theorem [**The Ratio Test**] Let $\sum a_n$ be a series and suppose that

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L.$$

Then,

- The series is **absolutely convergent** if $L<1$ (and therefore convergent!).
- The series is **divergent** if $L>1$ (or infinite).
- The test fails (or is inconclusive) if $L=1$ (no conclusion can be drawn about the convergence or divergence).

Theorem [**The Root Test**] Let $\sum a_n$ be a series and suppose that

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = L.$$

Then,

- The series is **absolutely convergent** if $L<1$ (and therefore convergent!).
- The series is **divergent** if $L>1$ (or infinite).
- The test fails (or is inconclusive) if $L=1$.

Definition [**Power series and their convergence**] A **power series in x or a power series about 0 or centered at 0** is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \ldots.$$

Remarks:

- The series converges **only** when $x=a$ corresponding to $R=0$.
- Use the **ratio test** to find the **radius** R and **interval** I of convergence. Note that I is **an open interval**.
- **Check the endpoints** separately to find the **complete** interval of convergence.
- $\exists$ **three** possibilities: $R=0$, $R=\infty$, or $R=c$ where c is a finite number.
- In the third case, there are **four** possibilities: $(a-R, a+R)$, $(a-R, a+R]$, $[a-R, a+R)$, $[a-R, a+R]$.

Definition [**Taylor and Mclaurin series**] The power series representation of $f(x)$ is called **Taylor series** and is given by

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k, \quad f^{(k)}(a) \doteq f^{(k)}(x)\Big|_{x=a}.$$

If $a=0$, then the Taylor series is called **Mclaurin series**.

Definition [**Taylor's formula/theorem**] If $f^{(k)} \in C[a,b]\ \forall k = 1, n$ and $f^{(n+1)} \in C(a,b)$ then

$$f(x) = \underbrace{\sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k}_{T_n(x)} + \underbrace{\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}}_{R_n(x)},$$

where $T_n(x)$ is the **Taylor polynomial of order** n and R_n is the **remainder**. The value c is between a and b.

Definition [**Remainder estimation theorem/Taylor's inequality**] If $\exists M > 0$ such that $|f^{(n+1)}(x)| \le M$ holds for $|x-a| \le d$, then the remainder term $R_n(x)$ of the Taylor series satisfies

$$\Big|R_n(x)\Big| \le M\frac{|x-a|^{n+1}}{(n+1)!}, \quad |x-a| \le d.$$

Definition [**Binomial series**] If k is any real number and $|x|<1$, then the series

$$(1+x)^k = \sum_{n=0}^{\infty}\binom{k}{n}x^n = 1 + \sum_{n=1}^{\infty}\binom{k}{n}x^n,$$

where $$\binom{k}{n} = \frac{k(k-1)(k-2)\dots(k-n+1)}{n!},$$

are called the **binomial series** and $\binom{k}{n}$ the **binomial coefficients**.

Definition [**Frequently used Mclaurin series**]

- $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ with $|x|<1$ or $I=(-1,1)$.
- $e^x = \sum_{n=0}^{\infty} \dfrac{x^n}{n!}$ with $I=\mathbb{R}$.
- $\sin x = \sum_{n=0}^{\infty} \dfrac{(-1)^n x^{2n+1}}{(2n+1)!}$ with $I=\mathbb{R}$.
- $\cos x = \sum_{n=0}^{\infty} \dfrac{(-1)^n x^{2n}}{(2n)!}$ with $I=\mathbb{R}$.
- $\log(1+x) = \sum_{n=1}^{\infty} \dfrac{(-1)^{n-1} x^{n}}{n}$ with $I=(-1,1]$.
- $\arctan x = \sum_{n=0}^{\infty} \dfrac{(-1)^n x^{2n+1}}{2n+1}$ with $I=[-1,1]$.

Definition [**Vector**] Given points $A(x_1,y_1,z_1)$ and $B(x_2,y_2,z_2)$ in $\mathbb{R}^3$, then $\overrightarrow{AB} = \langle x_2-x_1, y_2-y_1, z_2-z_1\rangle$ denotes the *arrow* or *vector* based at A with terminal point B.

Definition [**Equation of the sphere**] The equation of the sphere centered at (h,k,l) with radius r is given by

$$(x-h)^2 + (x-k)^2 + (x-l)^2 = r^2.$$

Theorem [**Length and distance**] Let $\mathbf{a} = \langle x_1, y_1, z_1\rangle$ be vector and let $A(x_2,y_2,z_2)$, $B(x_3,y_3,z_3)$ be points. Then:

1. The length of $\mathbf{a}$ is $|\mathbf{a}| = \sqrt{\mathbf{a}\cdot\mathbf{a}} = \sqrt{x_1^2+y_1^2+z_1^2}$.
2. The distance $d(A,B)$ from the point A to the point B is:

$$d(A,B) = |\overrightarrow{AB}| = \sqrt{(x_3-x_2)^2+(y_3-y_2)^2+(z_3-z_2)^2}.$$

Theorem [**Dot product and angle of vectors**] For nonzero vectors $\mathbf{a}, \mathbf{b}$

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta,$$

where $\theta \in [0,\pi]$ is the angle between the vectors. Furthermore, if $\mathbf{a} = \langle x_1,y_1,z_1\rangle$ and $\mathbf{b} = \langle x_2,y_2,z_2\rangle$, then

$$\mathbf{a}\cdot\mathbf{b} = x_1x_2 + y_1y_2 + z_1z_2.$$

Definition [**Direction angles and direction cosines**] The **direction angles** of a vector $\vec{a} = \langle a_1, a_2, a_3\rangle$ are the angles α, β and γ that $\vec{a}$ makes with the positive x-, y- and z- axes:

$$\cos\alpha = \frac{a_1}{|\vec{a}|}, \quad \cos\beta = \frac{a_2}{|\vec{a}|}, \quad \cos\gamma = \frac{a_3}{|\vec{a}|}.$$

Definition [**Vector and scalar projections**]

- The **vector projection** of $\mathbf{b}$ onto (in the direction of) $\mathbf{a}$ is
$$\text{proj}_{\mathbf{a}}\mathbf{b} = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{a}\cdot\mathbf{a}}\right)\mathbf{a} = \text{comp}_{\mathbf{a}}\mathbf{b}\,\frac{\mathbf{a}}{|\mathbf{a}|}.$$
- The **scalar projection** of $\mathbf{b}$ onto (in the direction of) $\mathbf{a}$ is
$$\text{comp}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|} = |\mathbf{b}|\cos\theta.$$

Definition [**Cross product**] The **cross product** $\mathbf{a}\times\mathbf{b}$ of vectors $\mathbf{a} = \langle a_1,a_2,a_3\rangle$ and $\mathbf{b} = \langle b_1,b_2,b_3\rangle$ can be calculated by:

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\mathbf{k}.$$

Note that $\mathbf{a}\times\mathbf{b}$ is $\perp$ to both $\mathbf{a}$ and $\mathbf{b}$ by construction.

Theorem The length of $\mathbf{a}\times\mathbf{b}$ is given by: $|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin(\theta)$, where $\theta\in[0,\pi]$ is the angle between $\mathbf{a}$ and $\mathbf{b}$. Also $|\mathbf{a}\times\mathbf{b}|$ is **area of the parallelogram** with sides $\mathbf{a}$ and $\mathbf{b}$. Note that it follows that area of the triangle with vertices $\langle 0,0,0\rangle$ and the position vectors $\mathbf{a}$ and $\mathbf{b}$ is: $|\mathbf{a}\times\mathbf{b}|/2$.

Theorem Let two vectors $\mathbf{a}$ and $\mathbf{b}$. Then:

- $\mathbf{a}\perp\mathbf{b}$ if $\mathbf{a}\cdot\mathbf{b} = 0$ (number).
- $\mathbf{a}/\!/\mathbf{b}$ if $\mathbf{a}\times\mathbf{b} = \mathbf{0}$ (vector!).

Definition [**Scalar triple product**] The **scalar triple product** of $\mathbf{a} = \langle a_1,a_2,a_3\rangle$, $\mathbf{b} = \langle b_1,b_2,b_3\rangle$ and $\mathbf{c} = \langle c_1,c_2,c_3\rangle$ can be calculated by the determinant:

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

Theorem The **volume of the parallelepiped** determined by the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ is the **absolute value** $V = |\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})|$.

Definition Given a point $P(x_0,y_0,z_0)$ and a vector $\mathbf{v} = \langle a,b,c\rangle$, the **vector equation** of the line L passing through P_0 in the direction of $\mathbf{v}$ is:

$$\mathbf{r}(t) = \mathbf{OP} + t\mathbf{v} = \langle x_0,y_0,z_0\rangle + t\langle a,b,c\rangle = \langle x_0+at, y_0+bt, z_0+ct\rangle.$$

The resulting equations:

$$x = x_0 + at,$$
$$y = y_0 + bt,$$
$$z = z_0 + ct,$$

are called the **parametric equations** for L.

Definition If t is eliminated from the parametric equations of the line L, we obtain the **symmetric equations** for L given by

$$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}.$$

Definition The distance D from a point S to a line that passes through P parallel to a vector $\mathbf{v}$ is given by

$$D = |\mathbf{PS}|\sin\theta = \frac{|\mathbf{PS}\times\mathbf{v}|}{|\mathbf{v}|}.$$

Definition

- The **plane** passing through the point $P(x_0,y_0,z_0)$ with normal vector $\mathbf{n} = \langle a,b,c\rangle$ is given by the following **vector** equation:
$$\mathbf{n}\cdot\langle x-x_0, y-y_0, z-z_0\rangle = 0,$$
where (x,y,z) denotes a **general point** on the plane.
- Equivalently, we have:
$$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$$
or
$$ax + by + cz = d \quad \text{with} \quad d = -(ax_0 + by_0 + cz_0).$$

- The **angle** between two intersecting planes with normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ is the angle between their normal vectors given by
$$\theta = \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1|\,|\mathbf{n}_2|}\right).$$
- The distance D from a point $P_1(x_1, y_1, z_1)$ to the plane $ax+by+cz+d=0$ with normal vector $\mathbf{n} = \langle a, b, c\rangle$ is given by
$$D = |\text{comp}_{\mathbf{n}}\mathbf{b}| = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2+b^2+c^2}}.$$

Definition [**Vector functions and properties**] Consider $\mathbf{r}(t)$ with $x = f(t)$, $y = g(t)$, and $z = h(t)$ and $t \in [a, b]$. Then:

- $\lim_{t\to t_0} \mathbf{r}(t) = \langle \lim_{t\to t_0} f(t), \lim_{t\to t_0} g(t), \lim_{t\to t_0} h(t)\rangle$.
- If $\mathbf{r}$ is continuous at $t = t_0$, then $\lim_{t\to t_0} \mathbf{r}(t) = \mathbf{r}(t_0)$.
- $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$ is continuous if $\mathbf{r}(t)$ is continuous $\forall t \in I$.
- The **velocity field** is $\mathbf{v}(t) = \mathbf{r}'(t) = \langle x'(t), y'(t), z'(t)\rangle$.
- The **acceleration field** is $\mathbf{a}(t) = \mathbf{r}''(t) = \langle x''(t), y''(t), z''(t)\rangle = \mathbf{v}'(t)$.
- The **indefinite integral** of $\mathbf{r}(t)$ wrt t is
$$\int \mathbf{r}(t)\,dt = \langle \int f(t)\,dt, \int g(t)\,dt, \int h(t)\,dt\rangle + \mathbf{c},$$
where $\mathbf{c}$ is an **arbitrary constant vector**.
- The **definite integral** of $\mathbf{r}(t)$ from a to b is
$$\int_a^b \mathbf{r}(t)\,dt = \langle \int_a^b f(t)\,dt, \int_a^b g(t)\,dt, \int_a^b h(t)\,dt\rangle,$$
- The **length** L of $\mathbf{r}(t)$ is the integral of the speed:
$$L = \int_a^b |\mathbf{r}'(t)| dt.$$

Definition [**Curvature and** TNB **frame**] Let a parametrization of C in $\mathbb{R}^3$ be $\mathbf{r} = \langle f(t), g(t), h(t)\rangle$. Then:

- The **unit tangent vector** is $\mathbf{T} = \dfrac{\mathbf{r}'}{|\mathbf{r}'|}$.
- The **unit normal vector** is $\mathbf{N} = \dfrac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}$.
- The **unit binormal vector** is $\mathbf{B} = \mathbf{T} \times \mathbf{N}$.
- The **curvature of the curve** is
$$\kappa \doteq \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3},$$
- The **speed** is given by $\dfrac{ds}{dt} = |\mathbf{r}'(t)|$, where s stands for the **arclength function**:
$$s(t) = \int_a^t |\mathbf{r}'(u)|\,du.$$

Definition The **acceleration vector** can be decomposed as
$$\mathbf{a} \doteq a_T\mathbf{T} + a_N\mathbf{N},$$
whence

- $a_T \doteq \dfrac{d}{dt}|\mathbf{r}'|$ is the **tangential** component of $\mathbf{a}$, and
- $a_N = \kappa|\mathbf{r}'|^2 = \dfrac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|} = \sqrt{|\mathbf{a}|^2 - a_T^2}$ is the **normal** component of $\mathbf{a}$.

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