Math 241 (Calculus IV) Theory Review Notes

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Definition [**Trace Curves**] The curves of intersection of a surface G with planes parallel to coordinate planes are called **trace curves**.

Definition [**Cylinder**] A cylinder is a surface that consists of all lines, called **rulings** that are parallel to a given line and pass through a given plane curve.

Definition [**Quadric Surfaces**] A quadric surface is the graph of a second-degree equation in three variables x, y, and z. Their general form is

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0,$$

where $A, B, \dots, J$ are all constants. Through translation and rotation, the above general form of a quadric surface can be cast into one of the two standard forms:

$$Ax^2 + By^2 + Cz^2 + J = 0, \quad \text{or} \quad Ax^2 + By^2 + Iz = 0.$$

Definition [**Multivariable Functions**] A function of two (or more) variables is a rule that assigns to each ordered pair (x, y) in a set D a **unique** real number denoted by $f(x, y)$. The set D is the domain of f, and its range is the set of values f takes on, that is:

$$\{f(x, y) : (x, y) \in D\}.$$

Definition [**Graph of a Function**] The graph of a function $f(x, y)$ is the set of all points (x, y, z) such that $z = f(x, y)$ and $(x, y) \in D$.

Definition [**Level Curves**] The **level curves** of a function f of two variables are the curves with equations

$$f(x, y) = k,$$

where k is a constant, and in the range of f.

Definition [**Continuity**] A function f of two variables is called **continuous at** (a, b) if

$$\lim_{(x,y)\mapsto(a,b)} f(x, y) = f(a, b).$$

We say f is **continuous on** D if f is continuous at **every point** (a, b) in D.

Definition [**Partial Derivatives**] If f is a function of two variables, its **partial derivatives** are the functions f_x and f_y defined by

$$f_x(x, y) = \lim_{h\to 0} \frac{f(x + h, y) - f(x, y)}{h}$$

$$f_y(x, y) = \lim_{h\to 0} \frac{f(x, y + h) - f(x, y)}{h}.$$

We have the following rule for calculating partial derivatives.

1. To find f_x, regard y as a constant and differentiate $f(x, y)$ with respect to x.
2. To find f_y, regard x as a constant and differentiate $f(x, y)$ with respect to y.

Definition [**Tangent Plane**] Suppose f has continuous partial derivatives. An equation of the tangent plane to the surface $z = f(x, y)$ at the point $P = (x_0, y_0, z_0)$ is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

Definition [**Linear Approximation**] The linear approximation of $f(x, y)$ at (a, b) is

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

Definition [**Total Differential**] The total differential for $z = f(x, y)$,

$$dz = f_x(x, y)\, dx + f_y(x, y)\, dy = \frac{\partial z}{\partial x}\, dx + \frac{\partial z}{\partial y}\, dy.$$

Theorem [**Chain Rule Case 1**] Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t. Then z is a differentiable function of t and

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

Theorem [**Chain Rule Case 2**] Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(s, t)$ and $y = h(s, t)$ are both differentiable functions of s and t. Then z is a differentiable function of t and

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}, \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}.$$

Theorem [**Implicit Differentiation**] Suppose that z is given implicitly as a function $z = f(x, y)$ by an equation $F(x, y, z) = 0$, i.e., $F(x, y, f(x, y)) = 0$ for all (x, y) in the domain of $f(x, y)$. Then:

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}.$$

Definition [**Gradient of scalar functions**] If f is a function of two variables x and y, then the **gradient** of f is the vector function ∇f defined by

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y)\rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}.$$

For $f(x, y, z)$, i.e., a function of three variables, we have

$$\nabla f = \langle f_x, f_y, f_z\rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}.$$

Theorem [**Directional Derivative**] If f is a differentiable function of x and y, then f has a directional derivative in the direction of any **unit** vector $\mathbf{u} = \langle a, b\rangle$ and

$$D_{\mathbf{u}} f(x, y) = \nabla f \cdot \mathbf{u} = f_x(x, y)a + f_y(x, y)b.$$

Similarly, for $f(x, y, z)$ and **unit** vector $\mathbf{u} = \langle a, b, c\rangle$, f has a directional derivative given by

$$D_{\mathbf{u}} f(x, y, z) = \nabla f \cdot \mathbf{u} = f_x(x, y, z)a + f_y(x, y, z)b + f_z(x, y, z)c.$$

Theorem [**Max Value of the directional derivative**] Suppose f is a differentiable function of two or three variables, then

$$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} = |\nabla f||u| \cos(\theta) = |\nabla f| \cos(\theta).$$

1. The maximum value of the directional derivative $D_{\mathbf{u}} f(\mathbf{x})$ is $|\nabla f(\mathbf{x})|$ and it occurs when $\mathbf{u}$ has the same direction as $\nabla f(\mathbf{x})$.
2. $D_{\mathbf{u}} f(\mathbf{x})$ is 0 in directions perpendicular to $\nabla f(\mathbf{x})$. Hence, for any constant k, at every point P of the level set surface (or curve) $f = k$, the gradient vector ∇f is perpendicular to the level set, and so $\nabla f(P)$ is a normal vector for the tangent plane (or tangent line) of the level set.

Theorem Suppose S is a surface determined as $F(x, y, z) = k$ for $k =$ constant. Then ∇F is everywhere normal or orthogonal to S.

Definition [**Critical Points**] A point (a, b) is called a **critical** point of $f(x, y)$ if $f_x(a, b) = f_y(a, b) = 0$. A critical point is a **saddle** point if the Hessian D defined in the next theorem is negative.

Theorem [**Second Derivative Test**] Suppose second partial derivatives of f are continuous on a disk with center (a, b), and $f_x(a, b) = 0$ and $f_y(a, b) = 0$ (that is, (a, b) is a critical point of f). Let

$$D = D(a, b) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2.$$

(a) If $D > 0$ and $f_{xx}(a,b) > 0$, then $f(a,b)$ is a local minimum.

(b) If $D > 0$ and $f_{xx}(a,b) < 0$, then $f(a,b)$ is a local maximum.

(c) If $D < 0$, then $f(a,b)$ is a saddle point.

Theorem [**Absolute Maximum**] To find the absolute maximum and minimum values of a continuous function f on a closed, bounded set D:

1. Find the values of f at the critical points of f in D.
2. Find the extreme values of f on the boundary of D.
3. The largest of the values from steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

Theorem [**Method of Lagrange Multipliers**] To find the maximum and minimum values of $f(x,y,z)$ subject to the constraint $g(x,y,z) = k$ (assuming that these extreme values exist and $\nabla g \neq \mathbf{0}$ on the surface $g(x,y,z) = k$):

1. Find all values of x, y, z, and λ such that
$$\nabla f(x,y,z) = \lambda \nabla g(x,y,z), \quad \text{and} \quad g(x,y,z) = k.$$
2. Evaluate f at all the points (x,y,z) that result from step 1. The largest of these values is the maximum value of f; the smallest is the minimum value of f.

Similar argument applies when f is subject to two constraints:

$$g(x,y,z) = k, \quad h(x,y,z) = c,$$

whence the equation we solve is:

$$\nabla f = \lambda \nabla g + \mu \nabla h,$$

for some numbers λ and μ.

Definition [**Iterated Integral**] The iterated integral of $f(x,y)$ on a rectangle $R = [a,b] \times [c,d]$ is

$$\int_a^b \int_c^d f(x,y)\, dy\, dx \quad \text{or} \quad \int_c^d \int_a^b f(x,y)\, dx\, dy.$$

One calculates the integral $\int_a^b \int_c^d f(x,y)\, dy\, dx$ by first calculating $A(x) = \int_c^d f(x,y)\, dy$, holding x constant, and then calculating $\int_a^b A(x)\, dx$ and similarly, for calculating the other integral.

Theorem [**Fubini's Theorem**] If f is continuous on the rectangle $R = \{(x,y) \mid a \leq x \leq b,\ c \leq y \leq d\}$, then

$$\int\int_R f(x,y)\, dA = \int_a^b \int_c^d f(x,y)\, dy\, dx = \int_c^d \int_a^b f(x,y)\, dx\, dy.$$

Theorem [**Type I and II regions**] If f is continuous on a type I region D such that

$$D = \{(x,y) \mid a \leq x \leq b,\ \ g_1(x) \leq y \leq g_2(x)\},$$

then
$$\int\int_D f(x,y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\, dy\, dx.$$

When D is a type II region:

$$D = \{(x,y) \mid c \leq y \leq d,\ h_1(y) \leq x \leq h_2(y)\},$$

then
$$\int\int_D f(x,y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\, dx\, dy.$$

Theorem [**Change to Polar Coordinates in a Double Integral**] If f is continuous on a polar rectangle R given by $0 \leq a \leq r \leq b$, $\alpha \leq \theta \leq \beta$, where $0 \leq \beta - \alpha \leq 2\pi$, then

$$\int_R \int f(x,y) dA = \int_\alpha^\beta \int_a^b f(r\cos(\theta), r\sin(\theta)) r\, dr\, d\theta.$$

Definition [**Center of Mass**] The coordinates $(\bar{x}, \bar{y})$ correspond to the center of mass of a lamina occupying the region D, and having density function $\rho(x,y)$ area

$$\bar{x} = \frac{M_y}{m} = \frac{1}{m} \iint_D x\rho(x,y)\, dA,$$
$$\bar{y} = \frac{M_x}{m} = \frac{1}{m} \iint_D y\rho(x,y)\, dA,$$

where the mass m is given by

$$m = \iint_D \rho(x,y)\, dA.$$

Definition [**Moments of Intertia**] Let a particle be of mass m. The moment of inertia (or second moment) of that particle about an axis is defined to be mr^2 where r is the distance from the particle to that axis. We now define:

- The **moment of inertia** of the lamina **about the x-axis** is given by: $I_x = \iint_D y^2 \rho(x,y)\, dA$.
- Similarly, the **moment of inertia** of the lamina **about the y-axis** is given by: $I_y = \iint_D x^2 \rho(x,y)\, dA$.
- The **moment of inertia about the origin**, also called the **polar moment of inertia** is given by $I_0 = I_x + I_y$.

Definition [**Fubini's Theorem for Triple Integrals**] If f is continuous on the rectangular box $B = [a,b] \times [c,d] \times [r,s]$, then:

$$\iiint_B f(x,y,z)\, dV = \int_r^s \int_c^d \int_a^b f(x,y,z)\, dxdydz.$$

Definition [**Type I, II, and III regions**] A solid region E is of Type I if it lies between the graphs of two continuous functions of x and y, i.e.,

$$E = \{(x,y,z) \mid (x,y) \in D, u_1(x,y) \leq z \leq u_2(x,y)\},$$

where D is the projection of E onto the xy-plane. For E being of Type I we have:

$$\iiint_E f(x,y,z)\, dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z) \right] dA.$$

A solid region E is of **Type II** if it is of the form of:

$$E = \{(x,y,z) \mid (y,z) \in D, u_1(y,z) \leq x \leq u_2(y,z)\},$$

where D is the projection of E onto the yz-plane. Thus, we have:

$$\iiint_E f(x,y,z)\, dV = \iint_D \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)\, dx \right] dA.$$

Finally, a solid region E is of **Type III** if it is of the form of:

$$E = \{(x,y,z) \mid (x,z) \in D, u_1(x,z) \leq y \leq u_2(x,z)\},$$

where D is the projection of E onto the xz-plane, and this way:

$$\iiint_E f(x,y,z)\, dV = \iint_D \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)\, dy \right] dA.$$

Definition [**Applications of Triple Integrals**] Let the density function of a solid that occupies the region E be $\rho(x,y,z)$ at any given point (x,y,z). Then:

- Its **mass** is given by $m = \iiint_E \rho(x,y,z)\, dV$.
- The **moments** about the three **coordinate planes** area

$$M_{yz} = \iiint_E x\rho(x,y,z)\, dV,$$
$$M_{xz} = \iiint_E y\rho(x,y,z)\, dV,$$
$$M_{xy} = \iiint_E z\rho(x,y,z)\, dV.$$

- The **center of mass** is located at $(\bar{x}, \bar{y}, \bar{z})$ where

$$\bar{x} = \frac{M_{yz}}{m}, \quad \bar{y} = \frac{M_{xz}}{m}, \quad \bar{z} = \frac{M_{xy}}{m}.$$

- If $\rho(x, y, z) = \text{const.}$, then the center of mass of the solid is called the **centroid** of E.
- The **moments of inertia** about the three coordinate axes are:

$$I_x = \iiint_E \left(y^2 + z^2\right) \rho(x, y, z)\, dV,$$
$$I_y = \iiint_E \left(x^2 + z^2\right) \rho(x, y, z)\, dV,$$
$$I_z = \iiint_E \left(x^2 + y^2\right) \rho(x, y, z)\, dV.$$

Definition [**Change to Cylindrical Coordinates in a Triple Integral**] If f is continuous on

$$E = \{(x, y, z) \mid (x, y) \in D, u_1(x, y) \leq z \leq u_2(x, y)\},$$

where D is given in polar coordinates:

$$D = \{(r, \theta) \mid \alpha \leq \theta \leq \beta, h_1(\theta) \leq r \leq h_2(\theta)\}.$$

then

$$\iiint_E f(x, y, z)\, dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\, r dz dr d\theta.$$

Definition [**Change to Spherical Coordinates in a Triple Integral**] If f is continuous on

$$E = \{(\rho, \theta, \phi) \mid a \leq \rho \leq b, \alpha \leq \theta \leq \beta, c \leq \phi \leq d\},$$

with $a \geq 0, \beta - \alpha \leq 2\pi, d - c \leq \pi$, then

$$\iiint_E f(x, y, z)\, dV = \int_c^d \int_\alpha^\beta \int_a^b f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\, \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi.$$

Definition [**Vector Fields**] Let D be a set in $\mathbb{R}^2$ (a plane region). A **vector field on** $\mathbb{R}^2$ is a function $\mathbf{F}$ that assigns to each point (x, y) in D a two-dimensional vector $\mathbf{F}(x, y)$.

On equally footing, let E be a subset of $\mathbb{R}^3$. A **vector field on** $\mathbb{R}^3$ is a function $\mathbf{F}$ that assigns to each point (x, y, z) in E a three-dimensional vector $\mathbf{F}(x, y, z)$.

Theorem [**(Another) Line Integral**] Suppose $f(x, y)$ is a continuous function on a differentiable curve $C(t)$, $C\colon [a, b] \to \mathbb{R}^2$. Then

$$\int_C f(x, y)\, ds = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

In the above formula, $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$ is the speed of $C(t)$ at time t. Similarly, if $C(t)$ is in $3D$, then

$$\int_C f(x, y, z)\, ds = \int_a^b f(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt.$$

Theorem

$$\int_C f(x, y)\, dx = \int_a^b f(x(t), y(t)) x'(t)\, dt$$
$$\int_C f(x, y)\, dy = \int_a^b f(x(t), y(t)) y'(t)\, dt.$$

Definition [**Line Integral**] Let $\mathbf{F}$ be a continuous vector field defined on a smooth curve C given by a vector function $\mathbf{r}(t), a \leq t \leq b$. Then the **line integral of F along C** is

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt = \int_C \mathbf{F} \cdot \mathbf{T}\, ds;$$

here, T is the unit tangent vector field to the parameterized curve C.

Theorem If C in $\mathbb{R}^3$ is parameterized by $\mathbf{r}(t)$ and $F = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$, then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P dx + Q dy + R dz.$$

Theorem [**Fundamental Theorem of Calculus**] Let C be a smooth curve given by the vector function $\mathbf{r}(t)$, $a \leq t \leq b$. Let f be a differentiable function of two or three variables whose gradient vector ∇f is continuous on C. Then

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$

Definition

1. A curve $\mathbf{r}\colon [a, b] \to \mathbb{R}^3$ (or $\mathbb{R}^3$) **closed** if $\mathbf{r}(a) = \mathbf{r}(b)$.
2. A domain $D \subset \mathbb{R}^3$ (or $\mathbb{R}^2$) is **open** if for any point p in D, a small ball (or disk) centered at p in $\mathbb{R}^3$ (in $\mathbb{R}^2$) is contained in D.
3. A domain $D \subset \mathbb{R}^3$ (or $\mathbb{R}^2$) is **connected** if any two points in D can be joined by a path contained inside D.
4. A curve $\mathbf{r}\colon [a, b] \to \mathbb{R}^3$ (or $\mathbb{R}^2$) is a **simple curve** if it doesn't intersect itself anywhere between its end points ($\mathbf{r}(t_1) \neq \mathbf{r}(t_2)$ when $a < t_1 < t_2 < b$).
5. An open, connected region $D \subset \mathbb{R}^2$ is a **simply-connected** region if any simple closed curve in D encloses only points that are in D.

Definition [**Conservative Vector Fields**] A vector field $\mathbf{F}$ is called **conservative** if it is the gradient of some $f(x, y)$; $f(x, y)$ is the **potential function** for $\mathbf{F}$.

Definition [**Path Independence**] If F is a continuous vector field with domain D, we say that the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is **independent of path** if $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$ for any two paths C_1 and C_2 in D with the same initial and the same terminal points.

Theorem If $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ is a conservative vector field, where P and Q have continuous first-order partial derivatives on a domain D, then throughout D we have

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

Theorem Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ be a vector field on an open simply-connected region D. Suppose that P and Q have continuous first-order derivatives and $\quad \dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x} \quad$ throughout D.

Then $\mathbf{F}$ is conservative.

Definition [**Positively Oriented Curves**] A simple closed parameterized curve C in $\mathbb{R}^2$ always bounds a bounded simply-connected domain D. We say that C is **positively oriented** if for the parametrization $\mathbf{r}(t)$ of C, the region D is always on the left as $\mathbf{r}(t)$ traverses C.

Theorem [**Green's Theorem**] Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. Then

$$\oint_C P dx + Q dy = \int_D \int \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$$

Theorem [**Green's Theorem and Area Formulas**] Green's Theorem gives the following formula for the area of D:

$$A = \oint_C x\, dy = -\oint_C y\, dx = \frac{1}{2} \oint_C x\, dy - y\, dx.$$

Definition [**Curl**] If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field in $\mathbb{R}^3$, and the partial derivatives of P, Q, and R all exist, then the **curl** of $\mathbf{F}$ is the vector field on $\mathbb{R}^3$ defined by:

$$\operatorname{curl} \mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}.$$

Theorem [**The curl of a gradient vector**] If $f : \mathbb{R}^3 \mapsto \mathbb{R}$ is a function of three variables that has continuous second-order partial derivatives, then

$$\operatorname{curl}(\nabla f) = \nabla \times (\nabla f) = \mathbf{0}.$$

Definition [**Div**]
If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field on $\mathbb{R}^3$, and P_x, Q_y, and R_z exist, then the **divergence** of $\mathbf{F}$ (abbreviated as **div** of $\mathbf{F}$) is the function of three variables defined by

$$\operatorname{div} \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

Theorem [**Divergence and Curl**]
If $\mathbf{F} = \langle P, Q, R \rangle$ is a vector field on $\mathbb{R}^3$, and P, Q, and R have continuous second-order partial derivatives, then

$$\operatorname{div} \operatorname{curl} \mathbf{F} = \nabla \cdot (\nabla \times \mathbf{F}) = 0.$$

Theorem [**Vector Forms of Green's Theorem**]

- First vector form:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D (\operatorname{curl} \mathbf{F}) \cdot \mathbf{k} \, dA.$$

- Second vector form:

$$\oint_C \mathbf{F} \cdot \eta \, ds = \iint_D \operatorname{div} \mathbf{F}(x, y) \, dA.$$

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